Non-negative discord strengthens the subadditivity of quantum entropy functions

Patrick J. Coles

Department of Physics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, USA

(Dated: Version of January 11, 2011)

The definition of quantum discord is generalized to allow for any concave entropy function, and its non-negativity strengthens the subadditivity condition for that entropy function. In a sense, this condition is *intermediate* in strength between subadditivity and strong subadditivity, hence called firm subadditivity, allowing one to further classify entropy functions based on whether they satisfy this intermediate condition. It is proven that the quadratic entropy $1 - \mathrm{Tr}(\rho^2)$ satisfies the firm subadditivity condition, whereas some other subadditive Tsallis entropies are not firmly subadditive.

PACS numbers:

I. INTRODUCTION

The strong subadditivity is "one of the most important and useful results in quantum information theory" (p. 519 of [1]). This property of the von Neumann entropy leads to the intuitive notion that information does not increase upon discarding a subsystem or upon quantum operations, and it is helpful in proving certain entropic uncertainty relations [2, 3]. It is unfortunate that the von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the only entropy known to have this property, as it is not the easiest entropy to calculate due to the matrix logarithm. For example the quadratic (often called linear) entropy $S_Q(\rho) = 1 - \mathrm{Tr}\rho^2$ simply involves squaring the density operator rather than having to diagonalize it, making S_Q easier to work with.

Using the strong subadditivity property of S, one can show that the quantum mutual information between two systems a and b upper-bounds the Holevo quantity between a and b, e.g. Lemma 1 of [3], or for earlier formulations that state this result as the non-negativity of the quantum discord, see Proposition 2 of [4] or Theorem 1 of [5]. This result is interesting in that it is purely quantum-mechanical with no classical analog, since the quantum mutual information equals (is the same thing as) the Holevo quantity for classically-correlated systems [4]. We find it interesting enough to give it a name, *firm subadditivity* (FSA), motivated by the fact that it is stronger than the subadditivity (SA) property (see below), yet it can be proven using the strong subadditivity (SSA) property, so "firm" seems like an appropriate intermediate adjective.

In this article, we show that S is not the only entropy function that has the FSA property. Indeed the quadratic entropy S_Q is *firmly subadditive*. On the other hand, numerical counterexamples for particular Tsallis entropies (defined below) show that there are some entropies that are subadditive but not *firmly* subadditive. Thus, the FSA property provides a new means with which to classify quantum entropy functions.

The remainder of this article is organized as follows. Section II introduces the notation, e.g. defining a generalized quantum discord for any concave entropy function, and gives further background information. Theorem 2 of Section III gives four mathematically-equivalent ways of stating the FSA property. For comparison, Theorem 3 of Section IV gives four conditions for S that are stronger than those in Theorem 2. The proof that S_Q satisfies FSA is given in Section V, followed by a discussion of the Tsallis entropies in Section VI. Some remarks on a condition, related to zero discord, that generalizes the additivity condition are found in Section VII, with concluding remarks in Section VIII.

II. NOTATION AND BACKGROUND

Consider the following quantum entropy functions,

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho),$$
$$S_R(\rho) = \frac{1}{1-q} \log \mathrm{Tr}(\rho^q), \quad 0 < q \leqslant 1,$$
$$S_T(\rho) = \frac{1}{1-q}[\mathrm{Tr}(\rho^q) - 1], \quad 0 < q,$$
$$S_Q(\rho) = 1 - \mathrm{Tr}(\rho^2), \qquad (1)$$

respectively the von Neumann, Renyi, Tsallis, and quadratic (often called linear) entropies, where S_R and S_T are equal to S in the limit $q = 1$, and S_T interpolates between S and S_Q as q goes from 1 to 2. All of these entropies are concave, $S_K(\sum_j p_j \rho_j) \geqslant \sum_j p_j S_K(\rho_j)$ for $0 < p_j < 1$ and $\sum p_j = 1$, for the ranges of q specified in (1). Here, and in what follows we use the notation S_K for any of the entropies in (1), dropping the K subscript when specifically referring to von Neumann S; or if one wishes think of S_K as a general entropy function [6], assumed to depend only on the non-zero spectrum of its input and assumed to be concave.

Given a bipartite quantum system with a density operator ρ_{ab}, partial traces ρ_a and ρ_b, one can define a generalized quantum mutual information:

$$S_K(a:b) = S_K(\rho_a) + S_K(\rho_b) - S_K(\rho_{ab}), \qquad (2)$$

which is non-negative for any ρ_{ab} if S_K is subadditive (SA):

$$S_K(\rho_a) + S_K(\rho_b) \geqslant S_K(\rho_{ab}), \quad S_K(a:b) \geqslant 0. \qquad (3)$$

Roughly speaking, thinking of entropy as the absence of information, (3) says that one gains (never loses) information by obtaining access to the joint system ab, relative to the information one has from only local access to a and b separately. The entropies S, S_Q, and S_T for $q \geqslant 1$ [7] are SA, but for $0 < q < 1$ neither S_T nor S_R are SA.

The SSA property of S has various equivalent statements (p. 519 of [1]), one being that for any tripartite state ρ_{abc}:

$$S(a\!:\!bc) \geqslant S(a\!:\!b), \qquad (4)$$

the quantum mutual information does not increase upon discarding a subsystem.

While $S(a\!:\!b)$ is a very useful global or holistic measure of the correlation between a and b, there is reason to believe that further insight can be gained from a measure that quantifies the presence of individual types of information [8] about a in b. For example for a two-qubit system ab described by density operator $\rho_{ab} = (|00\rangle\langle00| + |11\rangle\langle11|)/2$, the z-information (0 or 1) about a is perfectly contained in b while the x-information ($+$ or $-$) about a is completely absent from b. Precisely speaking, a type of information $P_a = \{P_{aj}\}$ about a is a projective decomposition of the identity $I_a = \sum_j P_{aj}$ on $\mathcal{H}_a$, as the P_{aj} represent mutually exclusive properties analogous to a classical sample space. However this notion can be extended [3] to POVMs, i.e. where the P_{aj} are positive operators not necessarily projectors, provided the P_{aj} are interpreted (physically) as orthogonal projectors on an enlarged Hilbert space, a Naimark extension. We employ the symbol $N_a = \{N_{aj}\}$ for a rank-1 POVM, i.e. a POVM in which all N_{aj} are rank-1 operators.

The example in the previous paragraph is an extreme case; in general one would like to quantify *partial* information, for which one could use [3] a Holevo quantity:

$$\chi_K(\{p_j, \rho_j\}) = S_K\Big(\sum_j p_j \rho_j\Big) - \sum_j p_j S_K(\rho_j), \qquad (5)$$

which measures the distinguishability of the distinct ρ_j, and is non-negative and equal to zero if and only if all ρ_j are identical, due to the (strict) concavity of all entropies in (1). As applied to the ensemble $\{p_j, \rho_{bj}\}$ generated by a POVM $P_a = \{P_{aj}\}$ on a and a bipartite state ρ_{ab},

$$p_j \rho_{bj} = \mathrm{Tr}_a(P_{aj}\rho_{ab}), \quad p_j = \mathrm{Tr}(P_{aj}\rho_a), \qquad (6)$$

the associated Holevo quantity

$$\chi_K(P_a, b) = S_K(\rho_b) - S_K(\rho_b|P_a), \qquad (7)$$

quantifies the amount of the P_a type of information that is contained in b. In (7) and later on we shall use $|P_a$ to denote conditioning on the result of measurement P_a, for example,

$$S_K(\rho_b|P_a) = \sum p_j S_K(\rho_{bj}),$$
$$S_K(\rho_a|P_a) = \sum p_j S_K(\rho_{aj}),$$
$$S_K(a\!:\!b|P_a) = \sum p_j[S_K(\rho_{aj}) + S_K(\rho_{bj}) - S_K(\rho_{abj})], \qquad (8)$$

where one uses the p_j and ρ_{bj} from (6) here; $S_K(\rho_{aj}) = S_K(K_j\rho_a K_j^\dagger) = S_K(\rho_{bcj})$ and $S_K(\rho_{abj}) = S_K[(K_j \otimes I)\rho_{ab}(K_j^\dagger \otimes I)] = S_K(\rho_{cj})$ where $P_{aj} = K_j^\dagger K_j$, c is a system that purifies ρ_{ab}, $\rho_{bcj} = \mathrm{Tr}_a(P_{aj}\rho_{abc})/p_j$, and $\rho_{cj} = \mathrm{Tr}_a(P_{aj}\rho_{ac})/p_j$.

We shall also find it useful to define a generalized quantum discord [4][1]:

$$\delta_K(P_a, b) := S_K(a\!:\!b) - \chi_K(P_a, b). \qquad (9)$$

A useful identity (an immediate consequence of Theorem 3 of [3]), used in proving Theorem 2 below, is

$$\delta_K(N_a, c) = S_K(\rho_a) - \chi_K(N_a, b), \qquad (10)$$

where N_a is a rank-1 POVM and c is a system that purifies ρ_{ab}; so the quantity on the right-hand-side represents a discord if the purifying system is kept in mind.

III. FIRM SUBADDITIVITY

We define the FSA condition as follows.
Definition 1. An entropy is *firmly subadditive* or satisfies the *firm subadditivity* condition if for all bipartite systems ab (all dimensions), all ρ_{ab}, and all POVMs P_a on a,

$$S_K(a\!:\!b) \geqslant \chi_K(P_a, b). \qquad (11)$$

Proposition 1. For any concave entropy function S_K, the FSA condition implies and strengthens the SA condition; if S_K is firmly subadditive, then it is also subadditive. $\square$

The fact that (11) is stronger than (3) clearly follows from the fact that $\chi_K(P_a, b) \geqslant 0$ if S_K is concave. Note that by symmetry one can obviously interchange the roles of a and b in (11), and also an alternative way of writing (11) is

$$\delta_K(P_a, b) \geqslant 0, \qquad (12)$$

the quantum discord is non-negative. Furthermore, to ensure (11) it is sufficient to check that $S_K(a\!:\!b) \geqslant \chi_K(w_a, b)$ for all orthonormal bases w_a, since by Naimark

---

[1] Some authors take $\min_{P_a} \delta_K(P_a, b)$ as the discord, we shall do no such minimization in this article.

extension P_a is equivalent to a projective measurement P_A on a larger Hilbert space $\mathcal{H}_A$, with $S_K(a\!:\!b) = S_K(A\!:\!b)$ and by the concavity of S_K, $\chi_K(P_A, b) \leqslant \chi_K(w_A, b)$ for some basis w_A of $\mathcal{H}_A$.

Conceptually, one can think of FSA in several ways. One can look at it as a statement that the "quantum" correlations are bigger than the "classical" correlations, but admittedly this is ambiguous language. Somewhat more precisely, the left-hand-side of (11) is a global or holistic measure of the correlation between a and b, whereas the right-hand-side quantifies a *single type* of information about a in b, so FSA says the global measure is bigger than the single-type measure. Alternatively, it says that the amount of information one gains by obtaining access to the joint system ab, relative to the local information in a and b separately, is not less than the amount that a single type of information about a is present in b.

The following theorem gives three other conditions that are mathematically equivalent to (11), for example, the Holevo quantity measuring the P_a information in b is bounded-above by the entropy of ρ_a (part ii), and the Holevo quantity is non-increasing under the action of a quantum channel for an input ensemble of pure states (part iv). As written below, these inequalities show that, if S_K is FSA, then this immediately implies some possibly useful bounds on the Holevo quantity χ_K associated with S_K.

Theorem 2. For any concave entropy function S_K depending only on the (non-zero) spectrum of its input, the following four conditions are equivalent, each implying the other three:

(i) The entropy S_K is firmly subadditive; i.e. for any bipartite state ρ_{ab} and any POVM P_a on a,

$$\chi_K(P_a, b) \leqslant S_K(a\!:\!b). \tag{13}$$

(ii) For any bipartite state ρ_{ab} with $\rho_a = \text{Tr}_b(\rho_{ab})$ and any POVM P_a on a, S_K satisfies:

$$\chi_K(P_a, b) \leqslant S_K(\rho_a). \tag{14}$$

(iii) For any tripartite *pure* state ρ_{abc} and *rank-1* POVM N_a on a, χ_K satisfies:

$$\chi_K(N_a, b) \leqslant \chi_K(N_a, bc). \tag{15}$$

(iv) For any quantum channel $\mathcal{E}$ and any ensemble of pure states $\{p_j, |\psi_j\rangle\langle\psi_j|\}$, χ_K satisfies:

$$\chi_K(\{p_j, \mathcal{E}(|\psi_j\rangle\langle\psi_j|)\}) \leqslant \chi_K(\{p_j, |\psi_j\rangle\langle\psi_j|\}). \tag{16}$$

Proof. We will show the equivalence of all four parts by showing (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iv). Starting from (iv), set the pure states $|\psi_j\rangle\langle\psi_j| = \text{Tr}_a[N_{aj}\rho_{abc}]/p_j = \rho_{bcj}$ to be the conditional density operators on bc associated with the rank-1 POVM N_a and pure state ρ_{abc}; this corresponds to setting $\chi_K(\{p_j, |\psi_j\rangle\langle\psi_j|\}) = \chi_K(N_a, bc)$. Now set $\mathcal{E}$ in (iv) to be the channel that partial traces over system c, i.e. $\rho_{bj} = \mathcal{E}(\rho_{bcj}) = \text{Tr}_c(\rho_{bcj})$, and the left-hand-side of (16) becomes $\chi_K(N_a, b)$, proving (iii).

By Theorem 2 of [3], $\chi_K(N_a, bc) = S_K(\rho_a)$, and by the concavity of S_K, $\chi_K(P_a, b) \leqslant \chi_K(N_a, b)$, where P_a, a general POVM, is a coarse-graining of some rank-1 POVM N_a, proving (ii). Now apply (ii) to N_a and add $\chi_K(N_a, c) - \chi_K(N_a, b) = S_K(\rho_c) - S_K(\rho_b)$, see Theorem 3 of [3], to both sides giving $\chi_K(N_a, c) \leqslant S_K(a\!:\!c)$; again invoking concavity $\chi_K(P_a, c) \leqslant \chi_K(N_a, c)$ and interchange the labels b and c to prove (i). To prove (iv), introduce a reference system a' of the same dimension as a; by Stinespring's dilation theorem any channel $\mathcal{E}(\rho_{a'}) = \text{Tr}_c(V\rho_{a'}V^\dagger)$ from a' to c can be constructed by an isometry V from a' to bc followed by a partial trace over b, and any pure-state ensemble $\{p_j, |\psi_j\rangle\langle\psi_j|\}$ on the input a' to $\mathcal{E}$ can be produced by the action of some rank-1 POVM N_a on a with an appropriate choice of a partially-entangled pure state (pre-probability) $|\Phi\rangle$ on aa', see Sec. IIB of [3] for the explicit construction. Now apply (i) to N_a and the tripartite pure state $|\Omega\rangle = (I_a \otimes V)|\Phi\rangle$ generated by acting V on the a' part of $|\Phi\rangle$, giving $\chi_K(N_a, c) \leqslant S_K(\rho_a)$ [after adding $\chi_K(N_a, c) - \chi_K(N_a, b) = S_K(\rho_c) - S_K(\rho_b)$ to both sides of (13).] But $\chi_K(N_a, c) = \chi_K(\{p_j, \mathcal{E}(|\psi_j\rangle\langle\psi_j|)\})$ and $S_K(\rho_a) = \chi_K(N_a, a') = \chi_K(\{p_j, |\psi_j\rangle\langle\psi_j|\})$, proving (iv). $\qquad\square$

IV. VON NEUMANN ENTROPY

For comparison, consider the following properties of S, each of which is stronger than its counterpart in Theorem 2, and each of which can be proved [9] by invoking both the strong subadditivity (4) and the generalized additivity [see (27) below] of S. All four properties below are equivalent in the sense that if a general entropy S_K satisfied one of them, it would automatically satisfy all of them.

Theorem 3. The von Neumann entropy S and its associated Holevo quantity χ satisfy the following four *equivalent* conditions:

(i) For any bipartite state ρ_{ab} and any POVM P_a on a:

$$\chi(P_a, b) \leqslant S(a\!:\!b) - S(a\!:\!b|P_a). \tag{17}$$

(ii) For any bipartite state ρ_{ab} with $\rho_a = \text{Tr}_b(\rho_{ab})$ and any POVM P_a on a:

$$\chi(P_a, b) \leqslant S(\rho_a) - S(\rho_a|P_a). \tag{18}$$

(iii) For any tripartite state ρ_{abc} and any POVM P_a on a:

$$\chi(P_a, b) \leqslant \chi(P_a, bc). \tag{19}$$

(iv) For any quantum channel $\mathcal{E}$ and any ensemble $\{p_j, \rho_j\}$:

$$\chi(\{p_j, \mathcal{E}(\rho_j)\}) \leqslant \chi(\{p_j, \rho_j\}). \tag{20}$$

Proof. Part (iii) was proven in [9] and we will show (iii)⇒(ii)⇒(i)⇒(iv)⇒(iii). For (ii), apply (iii) to a pure state ρ_{abc}, so $\chi(P_a, bc) = S(\rho_{bc}) - \sum_j p_j S(\rho_{bcj}) = S(\rho_a) - \sum_j p_j S(\rho_{aj})$. For (i), let c purify ρ_{ab}, add $\chi(P_a, c) - \chi(P_a, b) = S(\rho_c) - S(\rho_b) + S(\rho_b|P_a) - S(\rho_c|P_a)$ to both sides of (18), and interchange the labels b and c in the resulting inequality. For (i)⇒(iv), apply the same argument used in going from (i)⇒(iv) in the proof of Theorem 2, replacing the pure-state ensemble with an arbitrary ensemble $\{p_j, \rho_j\}$ and N_a with an arbitrary POVM P_a. Likewise for (iv)⇒(iii), let $\{p_j, \rho_j\}$ in (iv) be the ensemble generated by $p_j \rho_j = \mathrm{Tr}_a[P_{aj} \rho_{abc}]$, and set $\mathcal{E}$ in (iv) to be the channel that partial traces over c. □

V. QUADRATIC ENTROPY

The proof that S_Q is firmly subadditive relies on writing χ_Q in terms of the Hilbert-Schmidt distance:

$$D_{\mathrm{HS}}(\rho, \sigma) = \mathrm{Tr}[(\rho - \sigma)^2] \qquad (21)$$

and using a property of D_{HS} given in the following lemma.

Lemma 4. Let $\mathcal{E}$ be any quantum channel, let $\rho = |\phi\rangle\langle\phi|$ and $\sigma = |\psi\rangle\langle\psi|$ be *pure* states,
  (i) For $q \geqslant 1$, with $|X| = (X^\dagger X)^{1/2}$,

$$\mathrm{Tr}[|\rho - \sigma|^q] \geqslant \mathrm{Tr}[|\mathcal{E}(\rho) - \mathcal{E}(\sigma)|^q]. \qquad (22)$$

  (ii) Setting $q = 2$ gives

$$D_{\mathrm{HS}}(\rho, \sigma) \geqslant D_{\mathrm{HS}}[\mathcal{E}(\rho), \mathcal{E}(\sigma)]. \qquad (23)$$

Proof. Write $\rho - \sigma = \lambda(|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|)$ where $\langle\alpha|\beta\rangle = 0$ and $0 \leqslant \lambda \leqslant 1$, and $\mathcal{E}(\rho) - \mathcal{E}(\sigma) = \lambda[\mathcal{E}(|\alpha\rangle\langle\alpha|) - \mathcal{E}(|\beta\rangle\langle\beta|)] = \lambda[A - B]$ where A and B are positive operators with orthogonal support $AB = 0$, and $\mathrm{Tr}A = \mathrm{Tr}B \leqslant 1$. Then $\mathrm{Tr}[|\rho - \sigma|^q] = 2\lambda^q \geqslant \lambda^q[(\mathrm{Tr}A)^q + (\mathrm{Tr}B)^q] \geqslant \lambda^q[\mathrm{Tr}(A^q) + \mathrm{Tr}(B^q)] = \lambda^q \mathrm{Tr}|A - B|^q = \mathrm{Tr}[|\mathcal{E}(\rho) - \mathcal{E}(\sigma)|^q]$. □

Theorem 5. The quadratic entropy $S_Q(\rho) = 1 - \mathrm{Tr}(\rho^2)$ satisfies the firm subadditivity condition and thus has all four properties given in Theorem 2.

Proof. Through simple algebra, rewrite χ_Q as:

$$\chi_Q(\{p_j, \rho_j\}) = S_Q(\rho) - \sum_j p_j S_Q(\rho_j) \qquad (24)$$

$$= \sum_{j, j' > j} p_j p_{j'} D_{\mathrm{HS}}(\rho_j, \rho_{j'}). \qquad (25)$$

By Lemma 4, $\sum_{j, j' > j} p_j p_{j'} D_{\mathrm{HS}}(|\psi_j\rangle\langle\psi_j|, |\psi_{j'}\rangle\langle\psi_{j'}|) \geqslant \sum_{j, j' > j} p_j p_{j'} D_{\mathrm{HS}}[\mathcal{E}(|\psi_j\rangle\langle\psi_j|), \mathcal{E}(|\psi_{j'}\rangle\langle\psi_{j'}|)]$, meaning $\chi_Q(\{p_j, |\psi_j\rangle\langle\psi_j|\}) \geqslant \chi_Q(\{p_j, \mathcal{E}(|\psi_j\rangle\langle\psi_j|)\})$, which is condition (iv) of Theorem 2, equivalent to the other three conditions of Theorem 2. □
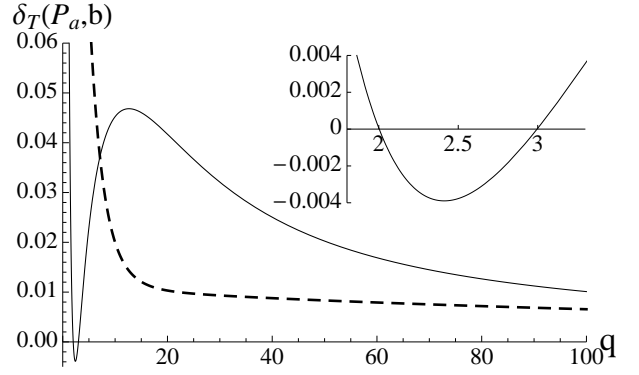

FIG. 1: Discord $\delta_T(P_a, b)$ as a function of q for two different choices (solid and dashed curves) of ρ_{ab} and P_a. The inset zooms in on the region where the discord goes negative for the solid curve.

VI. TSALLIS ENTROPY

Since both S and S_Q are FSA and both belong to the Tsallis entropy family, it is natural to ask whether other Tsallis entropies are FSA, in particular for exponent $q \geqslant 1$ in (1), for which S_T is subadditive. (Since S_T is not subadditive over the range $0 < q < 1$, it cannot be firmly subadditive over this range.) We have found numerical examples that violate the FSA condition for S_T over the range $2 + \epsilon < q < 3 - \epsilon$ for $\epsilon = .005$. This is illustrated in Fig. 1, which plots the Tsallis discord $\delta_T(P_a, b)$ as a function of q for two examples of ρ_{ab} and P_a; negative discord indicates a violation of the FSA condition. This shows that some entropies, S_T for $2 + \epsilon < q < 3 - \epsilon$, are subadditive but not firmly subadditve. It is not known whether S_T is FSA over the ranges $1 < q < 2$ and $q \geqslant 3$; our numerical searches in these ranges have not found any examples that violate the FSA condition.

VII. GENERALIZED ADDITIVITY

Thusfar we have discussed whether or not the discord δ_K is non-negative, consider the following remarks relevant to the conditions under which the discord is zero.

An entropy S_K is *additive* if

$$S_K(\rho_a \otimes \rho_b) = S_K(\rho_a) + S_K(\rho_b). \qquad (26)$$

Proposition 6. For some orthonormal basis $w = \{|w_j\rangle\}$ of $\mathcal{H}_a$, the condition

$$S_K(\sum_j p_j |w_j\rangle\langle w_j| \otimes \rho_{bj}) = S_K(\rho_a) + \sum_j p_j S_K(\rho_{bj}) \qquad (27)$$

generalizes (or strengthens) the additivity of S_K in that (27) implies (26); hence it may be called *generalized additivity.* □

This can be seen by writing $\rho_a \otimes \rho_b = \sum_j p_j |w_j\rangle\langle w_j| \otimes \rho_b$, where w is the eigenbasis of ρ_a, and applying (27)

to this state. Note that (27) is equivalent to saying that $\delta_K(w,b) = 0$ for $\rho_{ab} = \sum_j p_j |w_j\rangle\langle w_j| \otimes \rho_{bj}$. Since S_T is not additive (except for $q = 1$), it cannot satisfy (27). While S_R is additive, it does *not* satisfy (27) (again with the exception of $q = 1$, [4]). Thus, of the entropies in (1), S is unique in satisfying this generalized additivity condition.

VIII.   CONCLUSIONS

An information-theoretic condition, that the quantum mutual information upper-bounds the Holevo information, was proposed as a means to further classify quantum entropy functions. It is stronger than subadditivity, hence called the *firm subadditivity*. It was proven that the quadratic entropy has this property, showing that an entropy can be firmly subadditive without being strongly subadditive, while some other Tsallis entropies with exponent between 2 and 3 do not have this property, showing that an entropy can be subadditive without being firmly subadditive. Several mathematically equivalent expressions of the firm subadditivity were given in Theorem 2, providing bounds on, e.g. χ_Q, the quadratic Holevo quantity, which we anticipate to be useful due to its simplicity and its connection to the Hilbert-Schmidt distance in (25). The definition of discord was extended to entropy functions other than von Neumann's S; however, it is not yet clear whether these other discords will be useful for quantifying the non-classicality of correlations, since only S satisfies the generalized additivity condition in (27).

[1] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, 5th edition, 2000.
[2] Joseph M. Renes and Jean-Christian Boileau. Conjectured strong complementary information tradeoff. *Phys. Rev. Lett.*, 103:020402, 2009.
[3] P. J. Coles, L. Yu, V. Gheorghiu, and R. B. Griffiths. Information theoretic treatment of tripartite systems and quantum channels. eprint arXiv:1006.4859 [quant-ph].
[4] Harold Ollivier and Wojciech H. Zurek. Quantum discord: A measure of the quantumness of correlations. *Phys. Rev. Lett.*, 88:017901, 2001.
[5] A. Datta. A Condition for the Nullity of Quantum Discord. e-print arXiv:1003.5256 [quant-ph].
[6] Xinhua Hu and Zhongxing Ye. Generalized quantum entropy. *Journal of Mathematical Physics*, 47:023502, 2006.
[7] Koenraad M. R. Audenaert. Subadditivity of q-entropies for $q > 1$. *Journal of Mathematical Physics*, 48:083507, 2007.
[8] Robert B. Griffiths. Types of quantum information. *Phys. Rev. A*, 76:062320, 2007.
[9] Benjamin Schumacher, Michael Westmoreland, and William K. Wootters. Limitation on the amount of accessible information in a quantum channel. *Phys. Rev. Lett.*, 76:3452–3455, 1996.